# Synchronization of networks of chaotic units with time-delayed couplings

W. Kinzel,[1] A. Englert,[1] G. Reents,[1] M. Zigzag,[2] and I. Kanter[2]
[1]*Institute for Theoretical Physics, University of Würzburg, 97074 Würzburg, Germany*
[2]*Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel*
(Received 12 December 2008; revised manuscript received 6 March 2009; published 8 May 2009)

A network of chaotic units is investigated where the units are coupled by signals with a transmission delay. Any arbitrary finite network is considered where the chaotic trajectories of the uncoupled units are a solution of the dynamic equations of the network. It is shown that chaotic trajectories cannot be synchronized if the transmission delay is larger than the time scales of the individual units. For several models the master stability function is calculated which determines the maximal delay time for which synchronization is possible.

DOI: 10.1103/PhysRevE.79.056207 PACS number(s): 05.45.−a

A low-dimensional dynamical system can show irregular motion which is extremely sensitive to its initial state, hence in practice its motion cannot be predicted [1]. When two or more such chaotic units are coupled by exchanging signals of their variables they can synchronize: the motion of the whole network is still chaotic but the units may have identical common trajectories [2]. This is the case of complete synchronization which is considered in this contribution [3].

This counterintuitive phenomenon of chaos synchronization recently receives a lot of attention due to its neurobiological implications, its application on secure communication, its realizations on arrays of electronic circuits and lasers, and its mathematical fascination [4–7].

For almost all applications of chaotic networks, the coupling is time delayed due to the nonzero transmission time of the exchanged signals. Time delay can generate high-dimensional chaos (hyperchaos), an important example are semiconductor lasers where chaos is generated by time-delayed feedback [8–12]. Networks of chaotic units with delayed couplings can synchronize, as well [13–19].

Several methods have been developed to analyze the stability of the synchronized trajectory of chaotic networks [13–29]. Usually, the stability of the synchronization manifold is determined by transverse Lyapunov exponents. A powerful method combines the transverse Lyapunov exponents with eigenvalues of the coupling matrix. The resulting *master stability function* describes synchronization of any arbitrary network of given identical chaotic units [30].

In this contribution we calculate the master stability function of networks of chaotic units with time-delayed couplings. For the case, where the delay times of transmission are much larger than any characteristic time scales of the individual units, we conjecture that the chaotic trajectories of the individual units cannot be synchronized. This holds for any network including the case where the individual units contain self-feedback delays. For several models, chaotic flows as well as maps, we calculate the master stability function and determine the maximal delay time for which synchronization can occur.

Following Pecora and Caroll [30], we consider a set of $N$ identical units where $\mathbf{x}^i$ is the $m$-dimensional vector of dynamical variables of the $i$th unit. The isolated (uncoupled) dynamic is $\dot{\mathbf{x}}^i=\mathbf{F}(\mathbf{x}^i)$ for each node. For the coupling, the transmitted signal, we define a function $\mathbf{H}:\mathbb{R}^m\to\mathbb{R}^m$ which is identical for each unit. Thus, the dynamics of the network is defined as

$$\dot{\mathbf{x}}^i(t)=\mathbf{F}[\mathbf{x}^i(t)]+\sigma\sum_j G_{ij}\mathbf{H}[\mathbf{x}^j(t-\tau)]. \tag{1}$$

The matrix $G_{ij}$ determines the edges and weights of the network; $\sigma$ is the coupling strength. For the moment we consider a single delay time $\tau$, since it has been shown that multiple delay times destroy chaos synchronization [18]. We are interested in the question whether the chaotic trajectories of the single units can be synchronized by the second term of Eq. (1). Hence, we use $\Sigma_j G_{ij}=0$. Thus, the SM $\mathbf{x}^1=\mathbf{x}^2=\ldots=\mathbf{x}^N=\mathbf{x}$ with $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ is a solution of Eq. (1).

For a discrete time $t$ the corresponding equations for the iterated maps are

$$\mathbf{x}_t^i=\mathbf{F}(\mathbf{x}_{t-1}^i)+\sigma\sum_j G_{ij}\mathbf{H}(\mathbf{x}_{t-\tau}^j). \tag{2}$$

Pecora and Caroll have shown that the stability of the synchronization manifold can be analyzed with the eigenvalues $\gamma_k$, $k=0,1,\ldots,N-1$, of the coupling matrix $\mathbf{G}$. Linearizing Eq. (1) close to the SM gives a set of $mN$ coupled linear ODEs, but it is sufficient to consider the stability of $m$ ODEs [19,30],

$$\dot{\zeta}(t)=D\mathbf{F}[\mathbf{x}(t)]\zeta(t)+bD\mathbf{H}[\mathbf{x}(t-\tau)]\zeta(t-\tau). \tag{3}$$

$D\mathbf{F}$ and $D\mathbf{H}$ are the Jacobian functions evaluated at the synchronization manifold which is identical for each node. $\zeta$ is the $m$-dimensional vector which determines the stability of the $k$th eigenmode with $b=\sigma\gamma_k$. The master stability function is defined as the maximal Lyapunov exponent of Eq. (3).

By definition, the coupling matrix $\mathbf{G}$ has the eigenvalue $\gamma_k=0$ which belongs to the eigenmode tangential to the SM. Since each unit is supposed to be chaotic, Eq. (3) is unstable for $b=\sigma\gamma_k=0$, i.e., the maximal Lyapunov exponent $\lambda_0$ of the isolated units is positive. All other eigenvalues belong to perturbations away from the SM, i.e., they determine the stability of chaos synchronization. Thus, we want to know whether the second term of Eq. (3) can stabilize the network although the first term alone would lead to exponential explosion, i.e., whether the maximal Lyapunov exponent $\lambda_c$ of the coupled units can be negative although $\lambda_0$ is positive. When Eq. (3) is stable for *all* eigenvalues $\gamma_k\neq 0$, the network has a stable SM and the delayed coupling can synchronize the specific network. Note that the stability of Eq. (3) determines chaos synchronization for all possible networks, including delayed self-feedback.

The corresponding master stability equation for one-dimensional iterated maps is given by

$$\zeta_t = f'_{t-1}\zeta_{t-1} + bh'_{t-\tau}\zeta_{t-\tau}, \quad (4)$$

where the derivatives of the functions $f(x)$ and $h(x)$ are taken at the trajectory $x_t = f(x_{t-1})$.

The main result of this contribution is that the delay term of Eqs. (3) and (4) cannot damp the exploding local term when the delay time $\tau$ is sufficiently large. There are two arguments supporting this statement. First, the local term, without coupling, leads to $\zeta(t) \sim \exp(\lambda_0 t)$. Thus, when the system is unstable the delay term is exponentially small $\sim\exp(-\lambda_0\tau)$ compared with the local term, it has no influence on the stability for large $\tau$ [31]. This argument ignores fluctuations of the prefactors of Eqs. (3) and (4) which act like random multiplicative noise. In the limit of large delay, these fluctuations are uncorrelated, therefore they cannot counteract against the exponentially diverging perturbation driven by the local term of Eqs. (3) and (4).

We could not find a rigorous mathematical proof of our general conjecture, but our arguments are supported by the following analytical and numerical results of chaotic maps and flows. First, for an analytical solution, we consider the case of iterated maps, Eq. (4), for the Bernoulli shift $f(x) = h(x) = (\alpha)\text{mod } 1$ in the chaotic regime $\alpha > 1$. The local Lyapunov exponent $\lambda_0 = \ln \alpha$ is positive. The discontinuities of $f(x)$ do not influence the stability of Eq. (4), hence one has to solve the linear equation

$$\zeta_t = \alpha\zeta_{t-1} + b\alpha\zeta_{t-\tau}. \quad (5)$$

The ansatz $\zeta_t = (|c|e^{\imath\omega})^t$ yields $\tau$ solutions and the stability border $|c|_{\max} = 1$ is given by the equations

$$\alpha = \frac{\sin(\omega\tau)}{\sin[\omega(\tau-1)]}, \quad (6)$$

$$b = \frac{1}{\alpha}\cos(\omega\tau) - \cos[\omega(\tau-1)], \quad (7)$$

$$\alpha + b\alpha = 1, \quad (8)$$

with $\omega \in [0, \pi]$. These boundaries are shown in Fig. 1, for different values of delay $\tau$. Only for negative coupling constants the delay term can synchronize the chaotic network. The maximal possible delay time is obtained from Eq. (5) in the limit of $\omega \to 0$,

$$\tau_{\max} = \frac{\alpha}{\alpha - 1} = \frac{e^{\lambda_0}}{e^{\lambda_0} - 1}. \quad (9)$$

Thus, for large delay we find that synchronization can only occur if the delay is not much larger than $1/\lambda_0$, the characteristic time scale of the individual chaotic unit, or in other words, if the chaos is sufficiently weak.

For a more realistic model, the coefficients of Eq. (4) are not constant, but they fluctuate due to the underlying chaotic dynamics. Do fluctuations enhance synchronization? Our numerical simulation of the logistic map $f(x) = h(x) = rx(1-x)$ shows that the maximal value of the delay time $\tau_{\max}$ is not much larger than the Lyapunov time $1/\lambda_0$. Thus, fluctuations decrease the ability to synchronize.

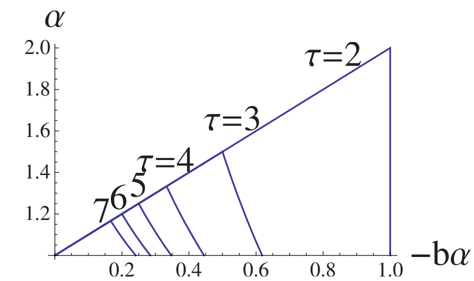


FIG. 1. (Color online) Regimes of stable complete synchronization. Inside the triangle the master stability function for a chaotic network is negative. $\alpha$ is the parameter of the Bernoulli shift with $\lambda_0 = \ln \alpha$ and $b$ is the rescaled coupling constant. With increasing delay time $\tau$ the region of synchronization shrinks.

The second support of our conjecture is shown for a network of coupled Roessler equations. We use the same parameters and coupling as in Ref. [30], $\dot{x} = -(y+x)$, $\dot{y} = x + 0.2y$, $\dot{z} = 0.2 + (x-0.7)z$, and $H(x,y,z) = (x,0,0)$. Figure 2 shows the results of the numerical simulation of the master stability Eq. (3).

For $\tau = 0$ we reproduce the results of Ref. [30]. With increasing delay the coupling region where chaos synchronization is possible shrinks up to $\tau_{\max} \approx 1$. For larger values of $\tau$ there is no network which can synchronize this Roessler system. The maximal delay is smaller then the Lyapunov time $\frac{1}{\lambda_0} \approx 10$, it is of the order of the time scale between the peaks of the variable $z(t)$.

Our examples for iterated maps and chaotic flows support the general conjecture: chaotic units cannot be synchronized when the coupling delay is larger then the local time scales. Similar results have been found in different contexts, namely, for chaos control and chaos anticipation: an unstable periodic orbit can only be stabilized and chaos can only be predicted if the delay time of the control terms is not much larger than the Lyapunov time [20,32,33].

We have considered a coupling matrix with $\Sigma G_{ij} = 0$ which leads to the synchronization of the isolated chaotic trajectories. But if this condition is released, then the synchronized trajectories are different from the isolated ones,

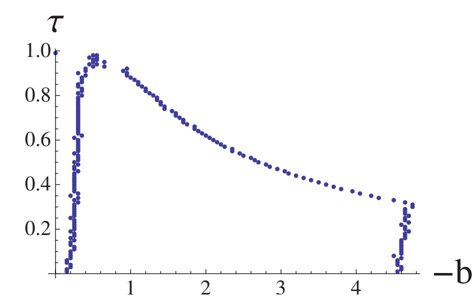


FIG. 2. (Color online) Regime of stable synchronization for a network of chaotic Roessler units. $b$ is the rescaled coupling constant and $\tau$ is the delay time of the transmitted signals.

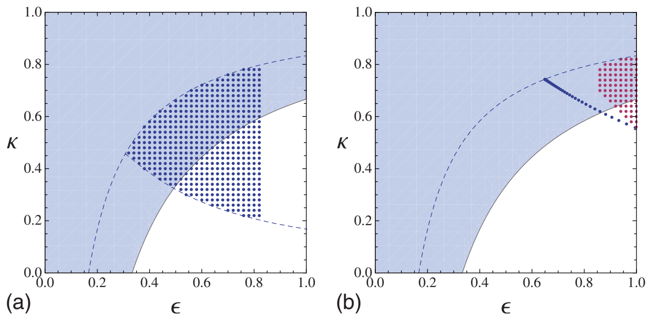


FIG. 3. (Color online) Analytical bounds of complete synchronization for two Bernoulli units with self-feedback. $\tau_f$ is the delay of the self-feedback and $\tau_c$ the one of the coupling. $\kappa$ determines the strength of the feedback and $\epsilon$ the one of the total delay terms, see Eq. (10). Above the solid line the isolated units with feedback are chaotic. Above the upper dashed line the two units cannot be synchronized, for any values of $\tau_f$ and $\tau_c$. The lower dashed line determines a bound for the border to complete synchronization. The dots are results from numerical simulations. $\tau_f=20$, $\alpha=1.5$, (a): $\tau_c=21$, (b): $\tau_c=60$.

and the local Lyapunov exponents $\lambda_0$ defined from the Jacobian $D\mathbf{F}$ evaluated at the new trajectory may become negative. This cannot occur for the Bernoulli shift, but for the logistic map we found that the delay term alone can generate negative values of $\lambda_0$ which leads to synchronized chaos.

One important example of the case $\Sigma_j G_{ij} \neq 0$ are two lasers coupled by their mutual laser beams [12]. This system is described by rate equations which have the structure of Eq. (1). The local term is nonchaotic, $\lambda_0 < 0$, and chaos is generated by the feedback terms $G_{ii}$ and coupling terms $G_{12}$ and $G_{21}$. Chaos synchronization has been achieved for extremely large delay times, namely, by transmitting light over 120 km [34–36]. The first laser has a feedback loop and drives a second laser without feedback. When the feedback time is identical to the coupling time complete synchronization without time shift is possible. But even when these delay times are different, the two chaotic trajectories can synchronize with a time lag. With short coupling times anticipated chaos has been found [33]. Our stability analysis shows that for this case synchronization can only occur if $\lambda_0$ of the second laser is negative. Chaotic units cannot be synchronized by unidirectional couplings, independent of the value of the coupling delay time.

For the case of bidirectional coupling, the previous statement is not valid. For example, when two lasers with feedback are mutually coupled one finds complete synchronization even when two isolated units with feedback are chaotic [37–41]. This holds even when the delay time of the coupling is different from the one of the feedback [42]. For the case with multiple coupling and feedback delays, and for networks with $\Sigma_j G_{ij} \neq 0$, we still do not know whether our conjecture is true, in general. But here we present an analytically solvable example which shows that for two chaotic systems with delayed feedback, synchronization is not possible if the coupling delay is sufficiently large.

We extend the analysis to a general network of identical chaotic units with feedback time $\tau_f$ which are coupled by exchanging signals with a delay time $\tau_c$. The corresponding equations for the Bernoulli map are

$$x_t^i = (1-\epsilon) f(x_{t-1}^i) + \epsilon \kappa f(x_{t-\tau_f}^i) + \epsilon (1-\kappa) \sum_{j \neq i} G_{ij} f(x_{t-\tau_c}^j), \tag{10}$$

with $\Sigma G_{ij}=1$. Now the matrix $G$ has an eigenvalue $\gamma_0=1$ which determines the Lyapunov exponent tangential to the SM. An stability analysis of Eq. (10) shows that the whole network is chaotic for all parameter values $\epsilon$ and $\kappa$. To compare with the case of lasers, we consider the case where the isolated units without feedback are not chaotic, i.e., $(1-\epsilon)\alpha < 1$. But with feedback a stability analysis similar to Eq. (4) shows that the uncoupled units are chaotic for

$$\kappa > \frac{1-\alpha(1-\epsilon)}{\alpha \epsilon}. \tag{11}$$

Note that our analysis shows that this border does not depend on the value of the feedback delay $\tau_f$. Adding the coupling term of Eq. (10) the chaotic units can be synchronized even if inequality (11) holds. However, we find the synchronization is not possible for

$$\kappa > 1 + \frac{1-\alpha}{\epsilon \alpha (1-\gamma_k)}, \tag{12}$$

where $\gamma_k$ are the eigenvalues of the coupling matrix $\mathbf{G}$ with $\gamma_k \neq \gamma_0 = 1$. Below border (12) synchronization depends sensitively on the values $\tau_f$ and $\tau_c$, as shown in Fig. 3 for the case of two interacting units ($\gamma_1=-1$). But our stability analysis shows that synchronization is not possible when

$$\tau_c > \frac{\alpha(1-\epsilon)(\gamma_k - 1)}{\gamma_k(\alpha-1)} + \frac{\alpha[1+\epsilon(\gamma_k-1)]-1}{\gamma_k(\alpha-1)} \tau_f \tag{13}$$

for all modes $\gamma_k \neq \gamma_0$. Thus, again we find that synchronization is not possible when the coupling delay is much larger than the time scales of the individual units. Note, however, that the prefactor of $\tau_f$ in Eq. (13) diverges for $\alpha \to 1$. Consequently, when the chaos of the isolated units is sufficiently weak, $\tau_c$ may become very large compared to $\tau_f$.

These analytic results for Bernoulli systems are supported by numerical simulations of the laser equations [42]. In the limit of large coupling delay, synchronization disappears when the lasers are chaotic due to the feedback delay.

[1] H. G. Schuster and W. Just, *Deterministic Chaos* (Wiley VCH, Weinheim, 2005).
[2] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[3] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization, a Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, 2001).
[4] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[5] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[6] W. Kinzel and I. Kanter, in *Handbook of Chaos Control*, 2nd ed., edited by E. Schöll and H. Schuster (Wiley-VCH, Weinheim, 2008).
[7] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
[8] V. Ahlers, U. Parlitz, and W. Lauterborn, Phys. Rev. E **58**, 7208 (1998).
[9] I. Fischer, R. Vicente, J. M. Buldu, M. Peil, C. R. Mirasso, M. C. Torrent, and J. García-Ojalvo, Phys. Rev. Lett. **97**, 123902 (2006).
[10] E. Klein, N. Gross, M. Rosenbluh, W. Kinzel, L. Khaykovich, and I. Kanter, Phys. Rev. E **73**, 066214 (2006).
[11] G. Kozyreff, A. G. Vladimirov, and P. Mandel, Phys. Rev. Lett. **85**, 3809 (2000).
[12] A. Uchida, F. Rogister, J. Garcfa-Ojalvo, and R. Roy, Prog. Opt. **48**, 203 (2005).
[13] M. Chen and J. Kurths, Phys. Rev. E **76**, 036212 (2007).
[14] F. M. Atay, J. Jost, and A. Wende, Phys. Rev. Lett. **92**, 144101 (2004).
[15] M. J. Bünner and W. Just, Phys. Rev. E **58**, R4072 (1998).
[16] M. G. Earl and S. H. Strogatz, Phys. Rev. E **67**, 036204 (2003).
[17] C. P. Li, W. G. Sun, and J. Kurths, Physica A **361**, 24 (2006).
[18] C. Masoller and A. C. Martí, Phys. Rev. Lett. **94**, 134102 (2005).
[19] M. Dhamala, V. K. Jirsa, and M. Ding, Phys. Rev. Lett. **92**, 074104 (2004).
[20] K. Pyragas, Phys. Rev. E **58**, 3067 (1998).
[21] P. G. Lind, J. A. C. Gallas, and H. J. Herrmann, Phys. Rev. E **70**, 056207 (2004).
[22] C. Zhou and J. Kurths, Phys. Rev. Lett. **96**, 164102 (2006).
[23] C. W. Wu, *Synchronization in Complex Networks of Nonlinear Dynamical Systems* (World Scientific, Singapore, 2007).
[24] I. Belykh, M. Hasler, M. Lauret, and H. Nijmeijer, Int. J. Bifurcat. Chaos **15**, 3423 (2005).
[25] I. Matskiv, Y. Maistrenko, and E. Mosekilde, Physica D **199**, 45 (2004).
[26] J. Jost and M. P. Joy, Phys. Rev. E **65**, 016201 (2001).
[27] Z. Duan, C. Liu, and G. Chen, Physica D **237**, 1006 (2008).
[28] C. Li, W. Sun, and J. Kurths, Phys. Rev. E **76**, 046204 (2007).
[29] M. S. Santhanam and S. Arora, Phys. Rev. E **76**, 026202 (2007).
[30] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[31] For periodic trajectories, this statement has been proven by S. Yanchuk and P. Perlikowski, Phys. Rev. E **79**, 046221 (2009).
[32] W. Just, E. Reibold, H. Benner, K. Kacperski, P. Fronczak, and J. Hołyst, Phys. Lett. A **254**, 158 (1999).
[33] H. U. Voss, Phys. Rev. E **61**, 5115 (2000).
[34] A. Argyris, D. Syvridis, L. Larger, V. Annovazzi-Lodi, P. Colet, I. Fischer, J. García-Ojalvo, C. R. Mirasso, L. Pesquera, and K. A. Shore, Nature (London) **438**, 343 (2005).
[35] A. L. Franz, R. Roy, L. B. Shaw, and I. B. Schwartz, Phys. Rev. E **78**, 016208 (2008).
[36] F. Zhang and P. L. Chu, J. Lightwave Technol. **21**, 3334 (2003).
[37] J. Kestler, E. Kopelowitz, I. Kanter, and W. Kinzel, Phys. Rev. E **77**, 046209 (2008).
[38] J. Kestler, W. Kinzel, and I. Kanter, Phys. Rev. E **76**, 035202(R) (2007).
[39] E. Klein, N. Gross, E. Kopelowitz, M. Rosenbluh, L. Khaykovich, W. Kinzel, and I. Kanter, Phys. Rev. E **74**, 046201 (2006).
[40] I. Kanter, N. Gross, E. Klein, E. Kopelowitz, P. Yoskovits, L. Khaykovich, W. Kinzel, and M. Rosenbluh, Phys. Rev. Lett. **98**, 154101 (2007).
[41] I. Kanter, E. Kopelowitz, and W. Kinzel, Phys. Rev. Lett. **101**, 084102 (2008).
[42] M. Zigzag, M. Butkovski, A. Englert, W. Kinzel, and I. Kanter, Europhys. Lett. **85**, 60005 (2009).